United States Golf Association

and

The Royal and Ancient Golf Club of St Andrews

Technical Description of the Pendulum Test

Abstract

In this document we describe the technical aspects of the pendulum tester. The central premise being that we wish to measure the flexibility of a club head in a repeatable, robust and reliable manner. We will describe the technical set up of the device and the rationale behind any modeling assumptions. We also give details of the data analysis, which is necessary to process the signals from the electronics.

1 Introduction

The objective of the pendulum test (shown in figure 1) is to give a more direct measurement of the flexibility of a club head than the current test for spring-like effect provides. It has been found that this test yields additional benefits over the current test in terms of portability, cost, time to perform as well as repeatability and resolution. The test generally consists of striking a steel mass with a spherical face (equipped with an accelerometer), against a club head and repeating this at several different impact velocities. It has been found that this procedure provides a measure of the flexibility of the club at high impact velocities.

The following document contains details of the model by which we determine the functional dependence of the velocity varying constituent of the characteristic time and a description of a model for the club/mass impact. The technical description of the test protocol and the details of the data analysis are given. Subsequently we give the analysis which is used within the code to determine a club's conformance.

2 A dynamic model for the pendulum test

In order to understand the dynamic properties of a club head it is essential that a realistic model is exploited, however it is also crucial that the model is not overly complicated. It is realised that it is insufficient to utilize a simple Hooke's law spring, and that the material and geometric effects can be incorporated via the use of a Hertzian spring.

The design of the test is based on the concept that we wish to use low velocity impacts to predict how the club will perform at higher velocities. To this end we divide the characteristic time into two constituents, being the velocity independent and dependent parts. Hence the characteristic time t_{char} is given by

$$t_{\text{char}} = \bar{t}_{\text{char}} + V_I^{-\frac{1}{5}}\mathcal{A}.$$

Figure 1: Photo of the test set up.

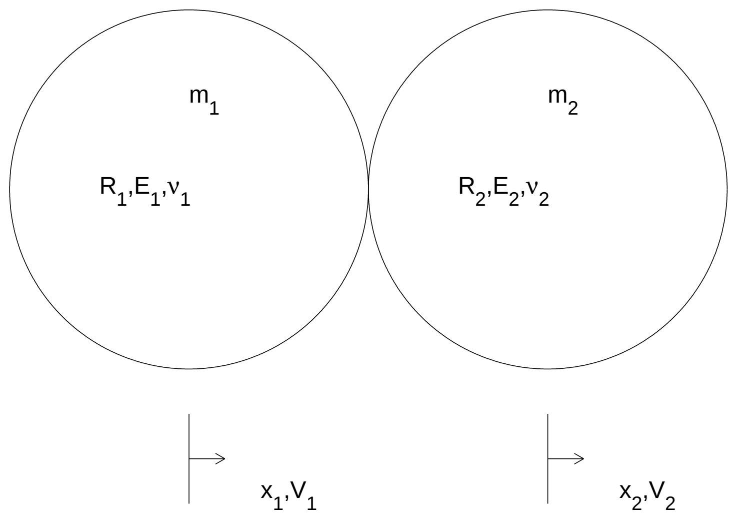

Figure 2: A schematic for the collinear impact of two spheres.

In this expression $\bar{t}_{\text{char}}$ is the characteristic time which reflects the flexibility of the club at high velocities (and it is this which will be used to judge conformance), while the latter part is the velocity dependent constituent. The functional dependence on the velocity of the test, V_I and an expression for the constant $\mathcal{A}$ are derived in the next section.

2.1 Hertzian contact - collinear impact of two spheres

We describe the collinear impact of spheres as presented in Johnson (1985)[1]. The spheres are taken to have radius of curvature R_j, Young's modulus E_j, mass m_j and Poisson's ratio of ν_j (for $j = 1$ and $j = 2$). In figure 2 we show the coordinate system, with the centers of the spheres at x_1 and x_2. We now consider the governing equations for the centers of the spheres. The force between the spheres while they are in contact is taken to be P and thus we have

$$\ddot{x}_1 = -\frac{1}{m_1}P \text{ and } \ddot{x}_2 = \frac{1}{m_2}P.$$

We note that the compression between the spheres is $\delta = x_1 - x_2$ (where the origins of these systems are at the respective centers of the spheres). Drawing on Johnson we note that the force P due to compression is $P = K\delta^{\frac{3}{2}}$, where

[1]K. L. Johnson 'Contact Mechanics' Cambridge University Press, 1985

$K = \frac{4}{3} R^{1/2} E^*$ with

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \text{ and } E^* = \frac{1}{\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}}.$$

In these expressions R is an effective radius of curvature and similarly E^* is the effective Young's modulus. We seek to solve the system subject to the initial conditions that $x_1 = x_2 = \dot{x}_2 = 0$ and $\dot{x}_1 = V_I$.

By subtracting the above equations for $\ddot{x}_1$ and $\ddot{x}_2$ we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1 - x_2) = -\frac{m_1 + m_2}{m_1 m_2} P$$

or written in terms of δ

$$\ddot{\delta} = -\frac{m_1 + m_2}{m_1 m_2} K \delta^{\frac{3}{2}}. \tag{1}$$

This can be integrated immediately with respect to time (by multiplying through by $\dot{\delta}$) to obtain

$$\frac{1}{2}\dot{\delta}^2 + \frac{m_1 + m_2}{m_1 m_2} \frac{2}{5} K \delta^{\frac{5}{2}} = \frac{1}{2} V_I^2. \tag{2}$$

In the above expression we have utilized the fact that $\delta = x_1 - x_2$ and consequently $\dot{\delta} = \dot{x}_1 - \dot{x}_2$, so that at $t = 0$ $\dot{\delta} = V_I$. Returning to the equation for $\ddot{x}_1$, eliminating P and noting that $\delta^{3/2}$ can be written in terms of $\ddot{\delta}$ we find that

$$\ddot{x}_1 = \frac{m_2}{m_1 + m_2} \ddot{\delta}$$

which can be integrated with respect to time to yield

$$\dot{x}_1 = \frac{m_2}{m_1 + m_2} \dot{\delta} + C.$$

Applying the initial conditions we find that

$$\dot{x}_1 = \frac{1}{m_1 + m_2} \left(m_2 \dot{\delta} + m_1 V_I \right). \tag{3}$$

The value of the velocity at the end of the collision (V_F) occurs when δ returns to zero (and there is no compression). This corresponds to $\dot{\delta} = -V_I$ (from equation (2)) and thus at this time

$$V_F = \frac{m_1 - m_2}{m_1 + m_2} V_I.$$

We are interested in the difference between the incoming and outgoing velocity, namely $V = V_F - V_I$. We now consider the generic problem in determining the value of δ at which $\dot{x}_1 = V_I + \lambda V$ ($\lambda = 0$ gives the initial velocity and $\lambda = 1$ the final velocity). In our example $\lambda = 0.05$ and $\lambda = 0.95$. A trivial rearrangement of the equation (3) gives the corresponding value of $\dot{\delta}$ as

$$\dot{\delta}_\lambda = \frac{(m_1 + m_2)(V_I + \lambda V) - m_1 V_I}{m_2} = (1 - 2\lambda)V_I.$$

We can now substitute this expression into (2) to obtain the corresponding value of δ, namely

$$\delta_\lambda = \left\{ \frac{5 m_1 m_2}{4K(m_1 + m_2)} 4\lambda(1 - \lambda) \right\}^{\frac{2}{5}} V_I^{\frac{4}{5}}.$$

We can now rearrange the equation (2) and integrate to show that the characteristic time is given by

$$\begin{aligned} \tilde{t}_{\rm char} &= \int_{\delta_{0.05}}^{\delta_{0.95}} \frac{\mathrm{d}\delta}{\sqrt{V_I^2 - \frac{m_1+m_2}{m_1 m_2}\frac{4}{5}K\delta^{\frac{5}{2}}}} \\ &= \int_{\delta_{0.05}}^{\delta_{0.5}} \frac{\mathrm{d}\delta}{\sqrt{V_I^2 - \frac{m_1+m_2}{m_1 m_2}\frac{4}{5}K\delta^{\frac{5}{2}}}} + \int_{\delta_{0.5}}^{\delta_{0.95}} \frac{\mathrm{d}\delta}{\sqrt{V_I^2 - \frac{m_1+m_2}{m_1 m_2}\frac{4}{5}K\delta^{\frac{5}{2}}}} \\ &= 2\int_{\delta_{0.05}}^{\delta_{0.5}} \frac{\mathrm{d}\delta}{\sqrt{V_I^2 - \frac{m_1+m_2}{m_1 m_2}\frac{4}{5}K\delta^{\frac{5}{2}}}}, \end{aligned}$$

where we have exploited the fact that δ_λ is symmetric about $\lambda = 1/2$. We now adopt a simple change of variables, so that

$$\delta = \left(\frac{5 m_1 m_2}{4K(m_1 + m_2)} \right)^{\frac{2}{5}} V_I^{\frac{4}{5}} x.$$

We note that the corresponding values of x_λ are given by simply $(4\lambda(1 - \lambda))^{2/5}$. Hence the characteristic time is

$$\tilde{t}_{\rm char} = \frac{2}{V_I^{\frac{1}{5}}} \left\{ \frac{5 m_1 m_2}{4K(m_1 + m_2)} \right\}^{\frac{2}{5}} \int_{(4\times 0.95\times 0.05)^{2/5}}^{1} \frac{\mathrm{d}x}{\sqrt{1 - x^{\frac{5}{2}}}}.$$

We note the functional dependence on the impact velocity. We also note that there is a dependence on mass. However, within any given scenario

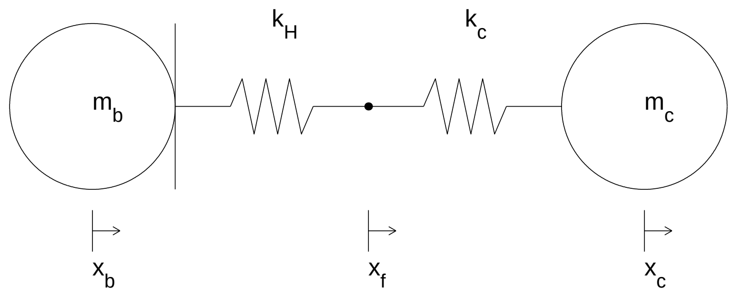

Figure 3: A schematic of the dynamic model of the club-mass interaction with a Hertzian contact.

the effective masses of the spheres are determined (that is by the mass of the club and the mass of the pendulum). Since we are proposing to use $\bar{t}_{\text{char}}$ to determine the conformance and the value of the constant $\mathcal{A}$ will be determined experimentally the presence of the mass in this expression is not a concern.

We note that when considering total time of impact (from $\lambda = 0$ to $\lambda = 1$) we find an analytical result in terms of the Beta function, however we exploit the 5% to 95% range to improve the repeatability of the evaluation. We should also mention that the model discussed in this section is pertinent to two spheres one of which impacts the other and they both are unconstrained. In the pendulum device the velocity we will measure is actually the difference between V_I and V_F, since we are in a frame moving with the pendulum's mass (specifically the accelerometer).

2.2 Model of the club/ball impact

We now proceed to discuss the model for the interaction of the club and the pendulum's mass. This draws on the previous section, using the Hertzian contact between the mass and the club. The basic model is depicted in figure 3. We define the coordinates x_b, x_f and x_c being representative of the mass, face and club respectively (with their origins as shown in the figure 3). We define the spring constant associated with the Hertzian spring k_H as $P/(x_b - x_f)$. Again following Johnson (1985) equation (4.23), we note that

$$(x_b - x_f)^3 = \frac{9P^2}{16R(E^*)^2}$$

where

$$R = \frac{1}{\frac{1}{R_b} + \frac{1}{R_c}} \text{ and } E^* = \frac{1}{\frac{1-\nu_b^2}{E_b} + \frac{1-\nu_c^2}{E_c}}.$$

Again R is an effective radius of curvature and similarly E^* is the effective Young's modulus. In these expressions R_b, ν_b and E_b are the radius of curvature, Poisson ratio and Young's modulus respectively for the mass (and similarly for the club with R_c, ν_c and E_c). Eliminating P from the above expressions we find that

$$k_H = \frac{4}{3} R^{\frac{1}{2}} E^* (x_b - x_f)^{\frac{1}{2}}.$$

We now give the governing equations for the system

$$m_b \ddot{x}_b = \begin{cases} k_H(x_f - x_b) & x_b > x_f \\ 0 & x_b < x_f, \end{cases}$$
$$m_c \ddot{x}_c = k_c(x_f - x_c);$$

recalling that each coordinate has a separate origin. Using a balance of forces at the point x_f we note that $k_H(x_b - x_f) = k_c(x_f - x_c)$. This can be manipulated to yield

$$x_f = \frac{k_H x_b + k_c x_c}{k_c + k_H}.$$

This expression can now be substituted into the above equations to give

$$m_b \ddot{x}_b = k_H \left[\frac{k_H x_b + k_c x_c}{k_c + k_H} - x_b \right] \text{ if } x_b > x_f,$$
$$m_c \ddot{x}_c = k_c \left[\frac{k_H x_b + k_c x_c}{k_c + k_H} - x_c \right],$$

where the expression for x_f also needs to be substituted into k_H. These equations can be solved effectively using a Runge-Kutta scheme to produce results whereby the model may be evaluated. In figure 4 we show three representative cases demonstrating the effect of changing the parameter k_c. This is useful for examining the role of various parameters and shows the way in which the Hertzian and Hookean springs interact.

We now proceed to discuss the manner in which we acquire the data.

3 The test protocol

Central to the test is the determination of a characteristic time which correlates with the current COR results for the clubs. Through an extensive set of tests it has been found that the most robust and repeatable measure is to:

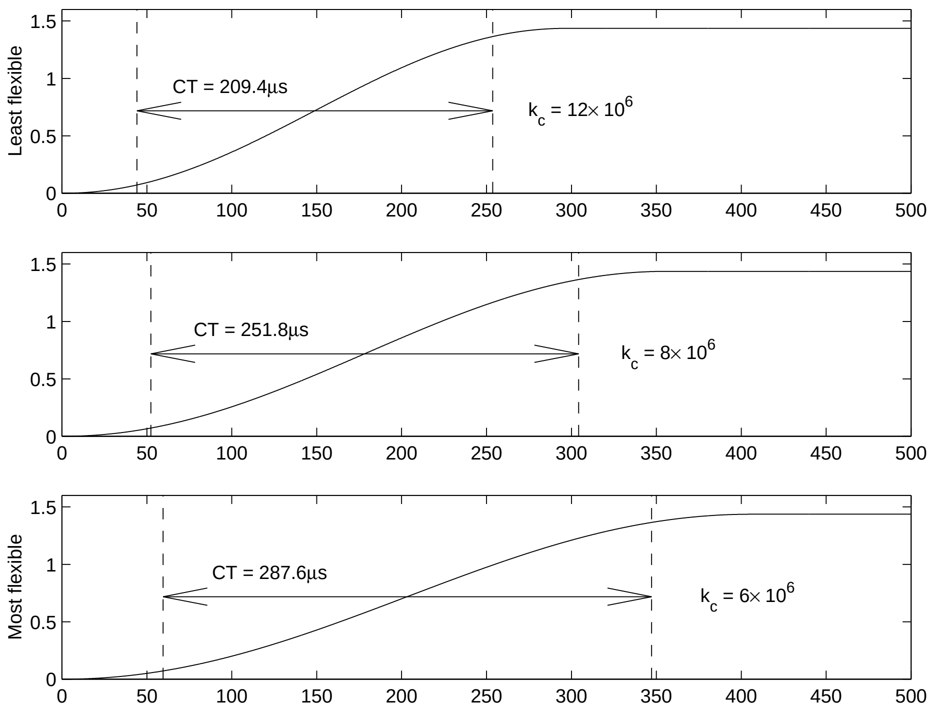

Figure 4: Plot of velocities showing characteristic times for three values of k_c (note that the higher the value of k_c the less flexible the club).

Filter the data from the scope and then integrate it to obtain the velocity of the accelerometer. The time taken for the velocity to rise from 5% to 95% of its maximum value is calculated. This test is repeated nine times (three times at three different settings on the pendulum). The acquired data is extrapolated to determine the effective characteristic time at large velocities.

We shall now describe the methods used for the data analysis. Further, we give details of how conformance is determined.

4 Data analysis

The mechanical device produces a signal $(t_i, \mathcal{S}_i)$ which spans 500μs. The indices of the points run from $i = 1$ to $i = N$. The symbol $\mathcal{S}$ is used for the voltage returning from the scope, which is representative of the acceleration. The recommended scope is a 12 bit device sampling at 50MHz (ADC212 www.picotech.com). At this point in this section we shall consider that the data has been extracted correctly, although we shall revisit the fidelity of the signal in due course. For the sampling rate of the scope we are using this gives $N = 3119$, with a corresponding time step of 0.16μs. An example of a signal from the scope is shown in figure 5. The accelerometer we are using has an intrinsic noise associated with oscillations of the crystal at 65-75KHz. In order to remove this and any other noise the Fourier spectrum of the data is calculated. Using the period of 500μs we have $L = 250\mu$s and the Fourier coefficients are

$$a_0 = \frac{2}{L}\int_{-L}^{L} \mathcal{S}(t)\,\mathrm{d}t$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \mathcal{S}(t)\cos\frac{n\pi(t-L)}{L}\,\mathrm{d}t,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \mathcal{S}(t)\sin\frac{n\pi(t-L)}{L}\,\mathrm{d}t.$$

Hence the reconstructed signal will take the form

$$\mathcal{S}_{N_m}(t) = a_0 + \sum_{i=1}^{N_m} a_n\cos\frac{n\pi(t-L)}{L} + b_n\sin\frac{n\pi(t-L)}{L}$$

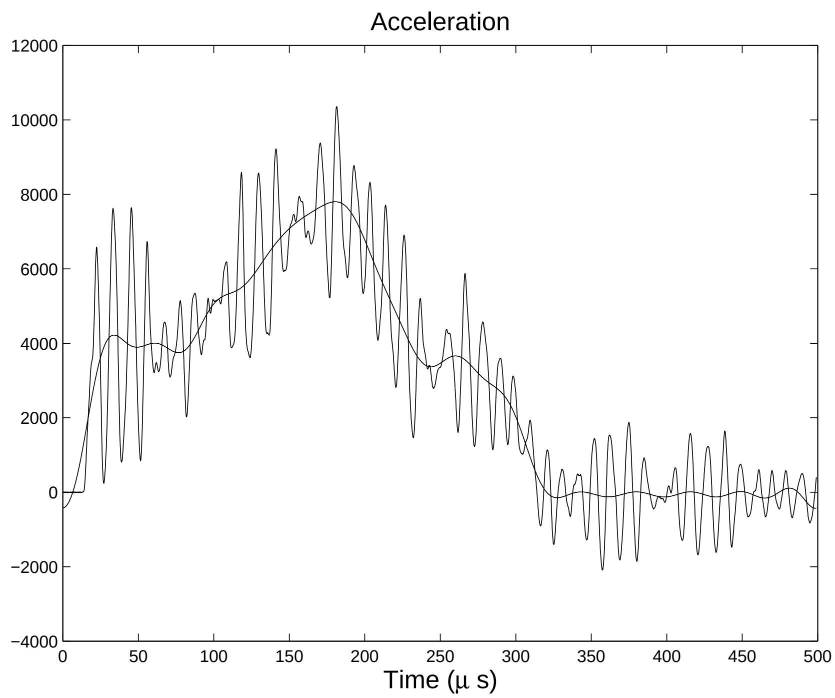

Figure 5: A typical example of the voltage output from the scope. This signal has 3119 points and the resolution is determined by the fact that the scope returns a 12 bit signal. We also include the corresponding filtered signal.

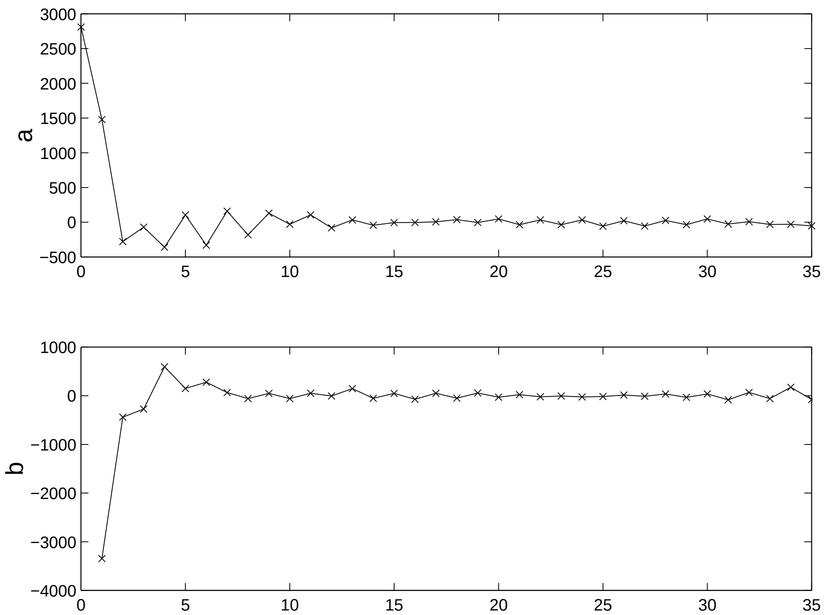

Figure 6: Fourier spectrum for a typical signal shown for the first 35 modes. Here we see some evidence of the noise (for $n \sim 34$).

where in the limit $N_m \to \infty$ we will recover the original signal. However, using a finite value of N_m we are able to remove the noise. It has been found suitable to use a value of 15. Extensive testing has been performed to verify that only small changes in the eventual answer are observed with variation in this number. In order to determine the Fourier coefficients it has been found to be sufficient to use a simple Trapezium method of numerical integration. The Fourier spectrum associated with the typical signal shown in figure 5 is shown in figure 6. We note that the acceleration is close to zero at both ends of the signal and consequently we do not encounter Gibb's phenomenon.

We wish to construct the velocity and this is done by integrating the signal $\mathcal{S}(t)$ from 0 to t, such that the initial velocity is zero. Hence we have

$$v_{N_m}(t) = a_0 t + \sum_{i=1}^{N_m} \frac{a_n L}{n\pi} \sin \frac{n\pi(t-L)}{L} - \frac{b_n L}{n\pi} \left(\cos \frac{n\pi(t-L)}{L} - (-1)^n \right), \quad (4)$$

where the term $(-1)^n$ comes from $\cos n\pi$. We are now in a position to manipulate this signal. Unsurprisingly, the resulting velocity is independent of whether the filtered signal is reconstructed and then integrated, or if it is constructed directly from equation (4). In figure 7 we show a representative velocity plot (which again corresponds to the acceleration signal shown in

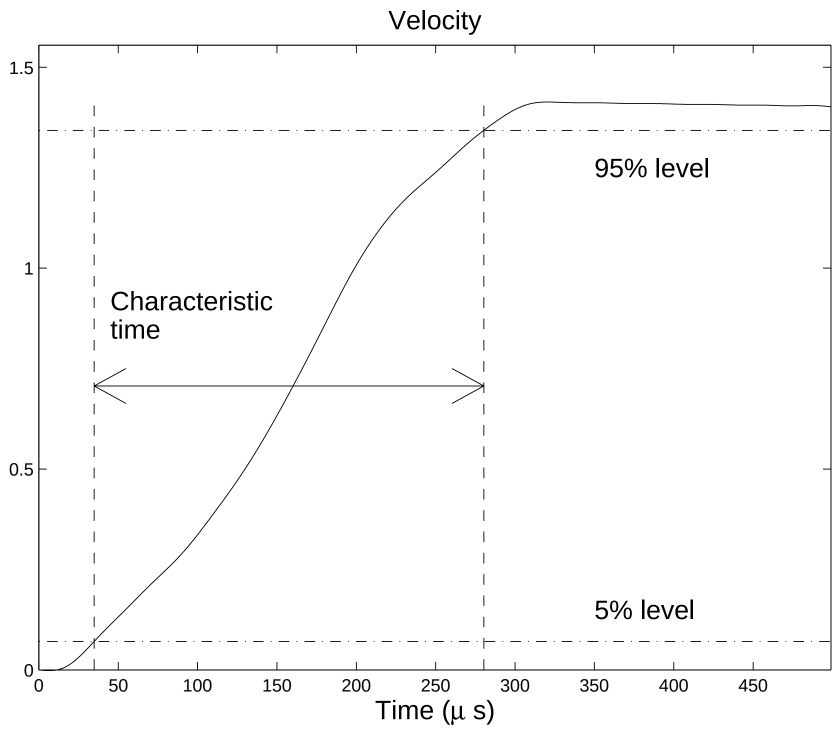

Figure 7: Sample velocity profile for an impact between a club and a ball, the units on the ordinate are ms^{-1}. Also shown are the times associated with the 5% and 95% points on the curve and a pictorial representation of the characteristic time.

figure 5).

In order to calculate a characteristic time for the club we determine the maximum velocity (simply by looking for the largest value of $v_N(t)$). We shall refer to this quantity as V (which gives a velocity representative of the impact). We then calculate the time for the velocity to rise from 5% of V to 95% of V. We exploit simple linear interpolation for this process with the filtered signal. This yields a value of t_{char}.

It is important that we predict how the characteristic time varies with V and as such the test is repeated for different values. This is achieved by dropping the pendulum from different heights. We shall now discuss how these points (V, t_{char}) are used to determine the conformance of a club.

5 Test procedure

The above procedure is repeated nine times to get the points $(V^{(i)}, {t_{\text{char}}}^{(i)})$ for $i = 1$ to $i = 9$. The test is based on the experiment being repeated three times at three different velocities. The test uses explicitly the change in velocity not the impact velocity that one might calculate from a potential energy calculation or direct measurement of the pre-impact velocity. We now exploit the model so that we shall determine a least squares fit of the data, such that

$$t_{\text{char}} = \hat{\alpha} + \hat{\beta} V^{-\frac{1}{5}}.$$

where

$$\hat{\beta} = \frac{\sum_{i=1}^{N_s} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N_s} (X_i - \bar{X})} \text{ and } \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X} = \bar{t}_{\text{char}}.$$

In the above expressions we have used $X = V^{-\frac{1}{5}}$ and $Y = t_{\text{char}}$. The standard error s_e is defined as

$$s_e = \sqrt{\frac{\sum_{i=1}^{N_s} (Y_i - \hat{Y}_i)^2}{N_s - 2}},$$

where $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$. We note that the $N_s - 2$ factor in the denominator is due to the fact that we have already estimated two parameters from the set of N_s points. The coefficient of variation of the data r^2 is given by

$$r^2 = \frac{\sigma_y^2 - s_e^2}{\sigma_y^2} \tag{5}$$

where σ_y is the sample standard deviation of Y (and in due course we will exploit the sample standard deviation of X, namely σ_x). These are defined as

$$\sigma_x^2 = \frac{1}{N_s - 1} \sum_{i=1}^{N_s} (X - \bar{X})^2 \text{ and } \sigma_y^2 = \frac{1}{N_s - 1} \sum_{i=1}^{N_s} (Y - \bar{Y})^2;$$

the factor of $N_s - 1$ is used to produce an unbiased estimate of the variance (since we are already estimating the mean). We have shown that the current COR test correlates with the characteristic time associated with very large V. Since it is impractical to test at these velocities we use extrapolation from the above tests. We realise that this can be prone to errors and hence it is crucial that large values of r^2 are obtained. We also exploit further analysis to identify confidence intervals.

Samples, N_s	DF	t_c (at 99%)
9	7	3.449
18	16	2.921

Table 1: Table of Student's t statistics at the 99% level.

5.1 Confidence intervals

We now seek to understand the likely variability in the extrapolated characteristic time and to this end we shall develop confidence intervals, which we define as:

Confidence interval The extent to which points on the regression line should vary for any of the inputs $V^{-1/5}$.

The extrema of the confidence interval are

$$\hat{Y} \pm s_e t_c \sqrt{\frac{1}{N_s} + \frac{(X - \bar{X})^2}{\sigma_x^2}}.$$

In these intervals t_c is the Student's t critical statistic with $N_s - 2$ degrees of freedom. The values used in the test are shown in Table 1. We also plan to incorporate measures into the software which will highlight any extraneous measurements, namely standardised residuals, leverages and their combination into Cook's distances.

6 Technical details of the protocol

A club will be tested in the prescribed manner (that is center hits) three times at each of the three velocities (corresponding to settings on the pendulum). These nine data points will be used to determine the line of best fit (in the manner described above). The 99% confidence interval will also be calculated for $y = 0$ (which corresponds to a large velocity impact).

If the confidence interval lies completely below the prescribed limit, then the club will be deemed to conform.

If the confidence interval lies completely above the prescribed limit, then the club will be deemed not to conform.

If the confidence interval includes the prescribed limit then a further nine hits (three at each velocity) will be taken and the extrapolated value

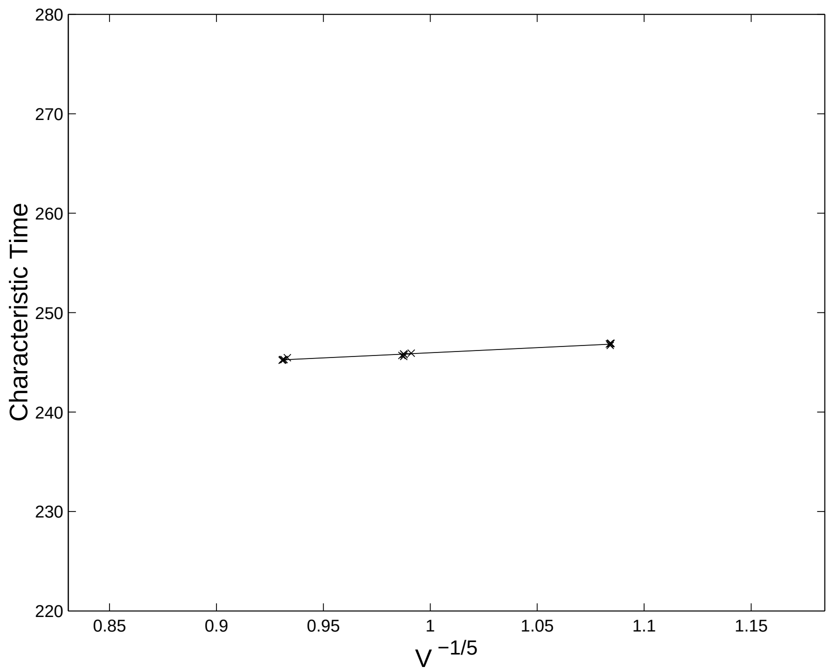

Figure 8: Variation of the characteristic time t_{char} with impact velocity V.

will be recalculated and this will be used to determine whether the club conforms or not.